Higher dimensional Kaluza-Klein Monopoles

Robert Mann[1] and Cristian Stelea[2]

[1]*Perimeter Institute for Theoretical Physics*
31 Caroline St. N. Waterloo, Ontario N2L 2Y5 , Canada

[1,2]*Department of Physics, University of Waterloo*
200 University Avenue West, Waterloo, Ontario N2L 3G1, Canada

Abstract

It is well known that the Kaluza-Klein monopole of Sorkin, Gross and Perry can be obtained from the Euclidean Taub-NUT solution with an extra compact fifth spatial dimension via Kaluza-Klein reduction. In this paper we consider Taub-NUT-like solutions of the vacuum Einstein field equations, with or without cosmological constant, in five dimensions and higher, and similarly perform Kaluza-Klein reductions to obtain new magnetic KK brane solutions in higher dimensions, as well as further sphere reductions to magnetic monopoles in four dimensions. In six dimensions we also employ spatial and timelike Hopf dualities to untwist the circle fibration characteristic to these spaces and obtain charged strings. A variation of these methods in ten dimensions leads to a non-uniform electric string in five-dimensions.

---

[1]E-mail: `rbmann@sciborg.uwaterloo.ca`
[2]E-mail: `cistelea@uwaterloo.ca`

1 Introduction

Although magnetic monopole solutions have been found in vacuum Kaluza-Klein theories since the 1980s [1, 2], it remains an outstanding problem to generalize these soliton solutions (here termed Kaluza-Klein monopoles, or KK monopoles) to include a cosmological constant. Given current theoretical interest in asymptotically (anti)-de Sitter spacetimes and recent experimental results that indicate that the universe does indeed possess a small positive cosmological constant, it is reasonable to pursue such an objective.

One such attempt was recently made by Onemli and Tekin [3]. They concluded that there is no five-dimensional static Kaluza-Klein monopole with cosmological constant. The metric ansatz they employed was tailored to describe a static 'Kaluza-Klein' monopole in an $AdS_2 \times AdS_3$ background. In the limit in which the cosmological constant λ tends to zero, the 'KK-AdS monopole' should reduce to the 'KK monopole' in flat space. Moreover, if the monopole charge tends to zero then the 'KK-AdS monopole' should reduce smoothly to the $AdS_2 \times AdS_3$ background. If we relax the requirement that the monopole solution be static it is easy to construct a time-dependent five-dimensional soliton that has all the desired properties. Two such solutions in five dimensions have been obtained in [3].

In the present paper we consider other possible extensions of Kaluza-Klein monopole solutions that admit a cosmological constant. The essential ingredient in the original Kaluza-Klein monopole construction is a Euclidean section of a Taub-NUT-like space; the 'trick' employed in [1, 2] to obtain the monopole solution is to lift this Euclidean section up to five-dimensions by adding a flat time coordinate and then to dimensionally reduce along the 'Euclidean time' direction from the Euclidean Taub-NUT section. However, in a presence of the cosmological constant it is not possible to use the above technique without introducing an explicit time dependence in the metric. Therefore, in order to obtain cosmological four-dimensional magnetic monopole solutions our strategy is to consider directly in five-dimensions the new cosmological Taub-NUT-like solutions[1], recently obtained in [5, 6] and perform a Kaluza-Klein compactification along the fifth dimension. The new feature of these solutions is that the four-dimensional dilaton acquires a potential term as an effect of the cosmological constant. However their asymptotics are not very appealing physically since they are not asymptotically flat or $(A)dS$ in the Einstein frame. Their metric description simplifies when considered in the string frame: for our explicit examples the four-dimensional metric in the string frame is very similar to the AdS form in the (r,t) sector, except for a

---

[1]It is also worth mentioning that, in a different context [4], some unexpected results were obtained for asymptotically AdS Taub-NUT spacetimes.

deficit of solid angle in the angular sector. Another interesting feature of the above constructions is that in five dimensions and in some of the higher dimensional examples the nut charge and the cosmological constant are intimately related by a constraint equation imposed by the equations of motion. This constraint makes it impossible to consider situations in which either the cosmological constant or the nut charge go to zero[2]. From this perspective the above monopole solutions are qualitatively distinct from their predecessors.

In higher than five dimensions we have more choices: we can consider solutions that are Ricci flat with different nut parameters or we can consider Taub-NUT like spaces that are constructed as circle fibrations over base spaces that have non-trivial topology. We also perform Kaluza-Klein (KK) reductions of the above solutions down to four dimensions, obtaining new magnetic monopole solutions. More specifically, in six and seven dimensions we have considered non-singular Ricci-flat solutions for which one can use the KK trick to obtain similar KK magnetic brane solutions for which the background spaces are Ricci flat Bohm spaces of the form $S^p \times S^q$ and generically have conical singularities. We considered their further reduction down to four dimensions on Riemannian spaces of constant curvature and specifically considered such reductions on spheres. In contrast with the KK procedure to untwist the $U(1)$-fibration, we have considered in six dimensions another method that is known to untwist the circle fibration, namely Hopf duality in string theory. We extended these duality rules to the case of a timelike Hopf-duality of the truncated six-dimensional Type II theories and applied them to generate charged string solutions in six-dimensions. By performing sphere reductions we obtained the corresponding four-dimensional solutions. In general, the presence of the cosmological constant in the higher dimensional theory induces a scalar potential for the Kaluza-Klein scalar fields. If the isometry generated by the Killing vector $\frac{\partial}{\partial z}$, which is associated with the circle direction on which we perform the reduction has fixed points, then the dilaton, which describes the radius of that extra-dimension, will diverge at the fixed point sets and the D-dimensional metric will be singular at those points. In certain cases we find that the dilaton field also diverges at infinity. Respectively this means that, physically, the space-time decompactifies near the KK-brane and at infinity; the higher-dimensional theory should be used when describing such objects in these regions.

The organization of this paper is as follows. We begin in section 2 by reviewing how the flat KK monopole can be obtained from the four dimensional Taub-nut solution. We also briefly discuss the KK monopole obtained by using the Euclidian Taub-bolt soliton. We next present in section 3 the new metric ansatz which is a solution of vacuum Einstein's equations

---

[2]See [7] for a way to evade this constraint in some special cases.

with cosmological constant in five dimensions and we perform a Kaluza-Klein reduction to obtain a new four-dimensional monopole solution. In the following sections we consider similar monopole solutions in higher dimensions and we also perform Kaluza-Klein sphere reductions to four dimensions. In six dimensions we apply spatial and timelike Hopf-dualities to generate new solutions. Section 5 concludes with some comments, and in appendices, for convenience, the KK reduction formulae and the T-duality rules which relate the truncated six-dimensional Type II theories are summarized.

2 The GPS magnetic monopole

We begin by reviewing the original magnetic monopole solution in 4 dimensions that arises as a Kaluza-Klein compactification of a 5 dimensional vacuum metric [1, 2]. The essential ingredient used in the monopole construction is a 4-dimensional version of the Taub-NUT solution, with Euclidean signature. The monopole solution is constructed as follows.

Start with the Euclidean form of the Taub-NUT solution [8]:

$$ds^2 = F_E(r)(d\chi - 2n\cos\theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)d\Omega^2$$

where

$$F_E(r) = \frac{r^2 - 2mr + n^2}{r^2 - n^2}$$

In general, the $U(1)$ isometry generated by the Killing vector $\frac{\partial}{\partial\chi}$ (that corresponds to the coordinate χ that parameterizes the fibre S^1) can have a zero-dimensional fixed point set (referred to as a 'nut' solution) or a two-dimensional fixed point set (correspondingly referred to as a 'bolt' solution). The regularity of the Euclidean Taub-nut solution requires that the period of χ be $\beta = 8\pi n$ (to ensure removal of the Dirac-Misner string singularity), $F_E(r = n) = 0$ (to ensure that the fixed point of the Killing vector $\frac{\partial}{\partial\chi}$ is zero-dimensional) and also $\beta F_E'(r = n) = 4\pi$ in order to avoid the presence of the conical singularities at $r = n$. With these conditions we obtain $m = n$, yielding

$$F_E(r) = \frac{r - n}{r + n}$$

Taking now the product of this Euclidean space-time with the real line, we obtain the following 5-dimensional Ricci flat metric:

$$ds^2 = -dt^2 + F_E(r)(d\chi - 2n\cos\theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)d\Omega^2$$

which solves the 5-dimensional vacuum Einstein equations.

If we perform now a Kaluza-Klein reduction along the coordinate χ (which is periodic with period $8\pi n$) we obtain the following 4-dimensional fields (with $\alpha = \frac{1}{2\sqrt{3}}$) [1, 2]

$$ds^2 = -F_E^{\frac{1}{2}} dt^2 + F_E^{-\frac{1}{2}}(r) dr^2 + F_E^{\frac{1}{2}}(r^2 - n^2) d\Omega^2$$
$$\mathcal{A} = -2n \cos\theta d\varphi, \qquad e^{\frac{\phi}{\sqrt{3}}} = F_E^{-\frac{1}{2}} \tag{2.1}$$

It is clear now that the metric is asymptotically flat. The above solution describes a magnetic monopole and its properties have been discussed in detail in [1, 2].

There are a few extensions of the above construction that we can consider. The obvious one to explore is the Taub-bolt solution in four-dimensions instead of the nut solution. In this case the Killing vector $\frac{\partial}{\partial\chi}$ has a two-dimensional fixed point set in the 4-dimensional Euclidean sector. The regularity of the solution is then ensured by the following conditions [9, 10, 11]: $F(r = r_b) = 0$ and $\frac{4\pi}{F'(r_b)} = \frac{8\pi n}{k}$ where k is an integer while the period of χ is now given by $\beta = \frac{8\pi n}{k}$, i.e. we identify k points on the circle described by χ.

It is easy to see that the above conditions are satisfied for $r_b = \frac{2n}{k}$ and $m = m_p = \frac{n(4+k^2)}{4k}$. We must demand that $r \geq r_b > n$, so that the fixed point set of $\frac{\partial}{\partial\chi}$ is not zero-dimensional; this in turn avoids the curvature singularity at $r = n$ and forces $k = 1$. Then the period of the coordinate χ is $8\pi n$ and for the bolt solution we obtain [9]:

$$F_E(r) = \frac{(r - 2n)\left(r - \frac{1}{2}n\right)}{r^2 - n^2} \tag{2.2}$$

As in the case of the nut solution, we take the product with the real line and obtain a metric in five-dimensions that is a solution of the vacuum Einstein field equations. Performing the Kaluza-Klein compactification along the χ direction yields (2.1), where now $F_E(r)$ is given by (2.2). The five-dimensional metric is regular everywhere for $r \geq 2n$. However, the four-dimensional solution obtained by Kaluza-Klein reduction, while asymptotically flat, is now singular at the location of the bolt $r = 2n$ where the dilaton field diverges as expected.

The physical interpretation of this solution was recently clarified by Liang and Teo [12] (see also [13]). It corresponds to a pair of coincident extremal dilatonic black holes with opposite magnetic charges. To see this we can use as a seed in the KK procedure the Euclidean rotating version of the bolt solution [14, 15]. We add a timelike flat direction in order to lift the solution to five dimensions, after which we reduce down to four dimensions. When $n \neq 0$ it has been shown in [12] that the above solution describes a pair of extremal dilatonic black holes carrying opposite but unbalanced magnetic charges and separated by a distance $2a$, a being the rotation parameter, which in this case serves as a measure of the proper distance between the black holes. In the limit $a \to 0$ we obtain the the solution

5

(2.1) and (2.2) which corresponds then to a pair of coincident monopoles that carry opposite unbalanced magnetic charges. The total magnetic charge of the system is n.

Since the above dihole has unbalanced charges it is not possible to introduce a background magnetic field to stabilize the system [12]. Consequently the solution is unstable and is expected to decay to a pure nut solution (the GPS soliton in this case) with total charge n.

3 Taub-Nut-dS/AdS spacetimes in 5 dimensions

In four dimensions the usual Taub-NUT construction corresponds to a $U(1)$-fibration over a two-dimensional Einstein space used as the base space. Usually taken to be the sphere S^2, it can also be the torus T^2 or the hyperboloid H^2. In five dimensions the corresponding base space is three dimensional and consequently the above construction is not straightforward. However, we can construct the five-dimensional Taub-NUT space as a sphere fibration over a factor space S^2 of the base space [5, 6]. The spacetimes that we obtain are not trivial in the sense that there is now a constraint on the possible values of the nut charge and the cosmological constant. Specifically, we cannot simultaneously set the nut charge and/or the cosmological constant to zero: there is no smooth limit in which one can obtain five dimensional Minkowski space in this way.

The ansatz that we shall use in the construction of these spaces is the following

$$ds^2 = -F(r)(dt - 2n\cos\theta d\varphi)^2 + F^{-1}(r)dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + r^2 dz^2 \quad (3.1)$$

The above metric will be a solution of the Einstein field equations with positive cosmological constant $\lambda = \frac{6}{l^2}$ provided

$$F(r) = \frac{4ml^2 - r^4 - 2n^2r^2}{l^2(r^2 + n^2)}$$

where the field equations impose the constraint $4n^2 = l^2$. Notice that for large values of r the function $F(r)$ takes negative values and r becomes effectively a timelike coordinate. We can write the solution directly in Euclidean signature (analytically continuing $t \to i\chi$ and $n \to in$):

$$ds^2 = F_E(r)(d\chi - 2n\cos\theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + r^2 dz^2 \quad (3.2)$$

where

$$F_E(r) = \frac{4r^4 - 2l^2r^2 + 16ml^2}{l^2(4r^2 - l^2)} \quad (3.3)$$

and we restrict the values of r to be greater than the largest root of F_E (if it has any) or than $l/2$, in order to preserve the metric signature. Since the analytic continuation of n

6

forces the continuation $l \to il$ for consistency with the initial constraint equation we obtain an Einstein metric with negative cosmological constant $\lambda = -\frac{6}{l^2}$.

In order to remove the usual Misner string singularity in the metric, we have to assume that the coordinate χ is periodic with some period β. Notice that for $r = n$ the fixed point of the Killing vector $\frac{\partial}{\partial \chi}$ is one dimensional (i.e. less than its maximal possible co-dimension) and we have a Taub-nut solution. However, for $r = r_b$, where $r_b > n$ is the largest root of $F_E(r)$, the fixed point set is three-dimensional (its maximal possible value); we shall refer to such solutions as Taub-bolt solutions.

As before, in order to have a regular nut solution we must ensure the same conditions on the periodicity of χ and the relationship $\beta F'_E(r = n) = 4\pi$. This leads to $m = \frac{l^2}{64}$; it is precisely for this value of the parameter m that the above solution becomes the Euclidean AdS spacetime in five-dimensions.

The bolt solutions likewise have the regularity conditions given in the previous section: the period of χ is $\beta = \frac{8\pi n}{k} = \frac{4\pi}{F'_E(r_b)}$, where k is an integer and $r = r_b > n$. These yield $r_b = \frac{kn}{2}$ and

$$m = m_b = \frac{k^2 l^2 (k^2 - 8)}{1024} \tag{3.4}$$

Setting $k \geq 3$ to ensure that $r_b > n$ (thereby also avoiding the curvature singularity at $r = n$), we obtain the following family of bolt solutions, indexed by the integer k:

$$ds^2 = F_E(r)(d\chi - l\cos\theta d\phi)^2 + F_E^{-1}(r)dr^2 + \left[r^2 - \left(\frac{l}{2}\right)^2\right](d\theta^2 + \sin^2\theta d\phi^2) + r^2 dz^2 \tag{3.5}$$

where

$$F_E(r) = \frac{256r^4 - 128l^2 r^2 - k^2 l^4 (k^2 - 8)}{64l^2 (4r^2 - l^2)} \tag{3.6}$$

However, for our purposes we need a five-dimensional solution that has Lorentzian signature. To obtain such a solution we shall analytically continue the coordinate $z \to it$. We then obtain the following five-dimensional space-time:

$$ds^2 = F_E(r)(d\chi - l\cos\theta d\phi)^2 + F_E^{-1}(r)dr^2 + \left[r^2 - \left(\frac{l}{2}\right)^2\right](d\theta^2 + \sin^2\theta d\varphi^2) - r^2 dt^2 \tag{3.7}$$

which is a solution of the vacuum Einstein field equations with a negative cosmological constant. Only when $k \geq 3$ (ensuring that $r_b = \frac{3l}{4} > \frac{l}{2}$) and $m = m_b$ from (3.4) is inserted into the metric function (3.3) will the solution (3.7) have no curvature singularities; otherwise there is a curvature singularity at $r = \frac{l}{2}$.

3.1 Monopole solutions in 4 dimensions

We are now ready to generate the magnetic monopole solutions in four dimensions using the same procedure as in the GPS case. Since the Taub-nut/bolt solution (3.1) is non-singular if we periodically identify the coordinate χ with period β, we can perform a Kaluza-Klein reduction along this direction. The Kaluza-Klein ansatz that we shall use is given in Appendix A, with $\alpha = \frac{1}{2\sqrt{3}}$.

Since our initial space-time is a solution of the Einstein field equations with non-zero cosmological constant, the field content in four dimensions will be given by the metric tensor $g_{\mu\nu}$, a magnetic one-form potential $\mathcal{A}$ and a scalar field ϕ with a non-trivial scalar potential $V(\phi)$. We obtain

$$ds^2 = -r^2 F_E^{\frac{1}{2}} dt^2 + F_E^{-\frac{1}{2}}(r)dr^2 + F_E^{\frac{1}{2}}\left(r^2 - \frac{l^2}{4}\right)d\Omega^2$$

$$\mathcal{A} = -2n\cos\theta d\varphi, \qquad e^{\frac{\phi}{\sqrt{3}}} = F_E^{-\frac{1}{2}} \tag{3.8}$$

where

$$F_E(r) = \frac{4r^4 - 2l^2 r^2 + 16ml^2}{l^2(4r^2 - l^2)}$$

This solution differs from the Kaluza-Klein GPS solution in terms of both the metric coefficients and by the fact that the scalar field ϕ has a potential of the exponential type

$$V(\phi) = -\frac{8}{l^2}e^{-\frac{\phi}{\sqrt{3}}}$$

indicating that our Kaluza-Klein dimensional reduction yields a massive scalar field.

To study the properties of the above 4-dimensional spaces, let us consider first the Taub-nut solution, which we have seen corresponds to the AdS solution in five-dimensions. For $m = \frac{l^2}{64}$ we obtain (with $n = \frac{l}{2}$):

$$F_E(r) = \frac{4r^2 - l^2}{4l^2}$$

and the five-dimensional metric becomes:

$$ds^2 = -(1 + \frac{R^2}{l^2})dt^2 + \frac{1}{1 + \frac{R^2}{l^2}}dR^2 + \frac{R^2}{4}\left[\frac{1}{l^2}(d\chi - \cos\theta d\varphi)^2 + d\theta^2 + \sin^2\theta d\varphi^2\right]$$

where we have defined $R = 4r^2 - l^2$ and rescaled the coordinates t and χ.

After Kaluza-Klein compactification we obtain the following four-dimensional fields:

$$ds^2 = -\frac{R}{2l}(1 + \frac{R^2}{l^2})dt^2 + \frac{RdR^2}{2l\left(1 + \frac{R^2}{l^2}\right)} + \frac{R^3}{8l}d\Omega^2$$

$$\mathcal{A} = -l\cos\theta d\varphi, \qquad e^{-\frac{\phi}{\sqrt{3}}} = \frac{R}{2l} \tag{3.9}$$

8

The four-dimensional metric has a curvature singularity at $R = 0$, where the scalar field also diverges. Its asymptotic structure is given by

$$ds^2 = -\frac{\tilde{R}^2}{l^2}dt^2 + \frac{4}{9}\left(\frac{l}{\tilde{R}}\right)^{\frac{4}{3}}d\tilde{R}^2 + \tilde{R}^2 d\Omega^2$$

where we have rescaled t and defined $\tilde{R}$ by $\tilde{R}^2 = \frac{R^3}{8l}$ and we can see that it is not asymptotically flat. However, one can check by computing some of the curvature scalars (like $R_{abcd}R^{abcd}$) that the above asymptotic metric is well-behaved at infinity and that it has a curvature singularity at $\tilde{R} = 0$. Notice however that at both $R = 0$ and $R \to \infty$ the dilaton is blowing up and the extra dimension opens up, which means that the physical description is effectively five-dimensional.

For the bolt solution we must take $k \geq 3$ and $m = m_p$, for which

$$F_E(r) = \frac{256r^4 - 128l^2r^2 - k^2l^4(k^2 - 8)}{64l^2(4r^2 - l^2)}$$

The compactified four-dimensional solution is given again by (3.8) and its asymptotic structure is the same as the one obtained from the five-dimensional AdS spacetime. Again the four-dimensional metric will have a curvature singularity at the bolt while the dilaton diverges at $r = r_b$ and at infinity.

Next, let us notice that in the string frame the situation changes as follows. The five-dimensional nut solution reduces to the following metric:

$$ds^2 = -(1 + \frac{R^2}{l^2})dt^2 + \frac{1}{1 + \frac{R^2}{l^2}}dR^2 + \frac{R^2}{4}\left[d\theta^2 + \sin^2\theta d\varphi^2\right]$$

While this metric resembles a four-dimensional AdS metric with cosmological constant $\lambda = -\frac{3}{l^2}$ in fact there is a deficit of solid angle as the area of the 2-sphere is πR^2 instead of $4\pi R^2$. This behavior is characteristic of a global monopole [16]. The above metric has a curvature singularity at the origin (the location of the monopole) and the dilaton field diverges both at origin and at infinity. The magnetic charge is computed using the formula [17]:

$$\frac{1}{4\pi}\int_{S^2} \mathcal{F} = l$$

Note that if we reduce directly the five-dimensional metric (3.1), which is time-dependent (outside the cosmological horizon the coordinate r is timelike) we obtain a time-dependent four-dimensional magnetic monopole solution.

One characteristic feature of the above constructions is that the nut charge and the cosmological constant are intimately related by a constraint equation imposed by the equations

9

of motion. This constraint makes it impossible for us to consider the cases in which either the cosmological constant or the nut charge go to zero. From this perspective the above monopole solutions are qualitatively distinct from their predecessors.

4 Higher dimensional magnetic monopoles

We now consider some of the higher dimensional Taub-NUT spaces constructed recently [5, 6]. In the Taub-NUT ansatz the idea is to construct such spaces as radial extensions of $U(1)$-fibrations over an even dimensional base endowed with an Einstein-Kähler metric. In a $(2k+2)$-dimensional Taub-NUT space the base factor over which one constructs the circle fibration can have at most dimension $2k$. However, it is not necessary to construct the circle fibration over the whole base space; in general one can consider factorizations of the base of the form $B = M \times Y$ in which M is endowed with an Einstein-Kähler metric while Y is an Einstein space with the metric g_Y. In these cases one can consider the $U(1)$-fibration only over the factor space M of the base and take then a warped product with the manifold Y. The ansatz is then given by

$$F^{-1}(r)dr^2 + (r^2 + N^2)g_M + r^2 g_Y - F(r)(dt + A)^2 \tag{4.1}$$

We now consider particular cases of this ansatz.

4.1 Six dimensional metrics

In six dimensions the base space is four-dimensional and we can use products of the form $M_1 \times M_2$ of two-dimensional Einstein spaces or we can use CP^2 as a four-dimensional base space over which to construct the circle fibrations. If we use products of two dimensional Einstein spaces then we can consider all the cases in which M_i, $i = 1, 2$ can be a sphere S^2, a torus T^2 or a hyperboloid H^2. The circle fibration can be constructed in these cases over the whole base space $M_1 \times M_2$ or just over one factor space M_i.

We shall consider first the case in which $M_1 = M_2 = S^2$ and assume that the $U(1)$ fibration is constructed over the whole base space $S^2 \times S^2$. Then the corresponding six-dimensional Taub-NUT solution is given by [5]

$$ds^2 = -F(r)(dt - 2n_1 \cos\theta_1 d\varphi_1 - 2n_2 \cos\theta_2 d\varphi_2)^2 + F^{-1}(r)dr^2$$
$$+(r^2 + n_1^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + (r^2 + n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2) \tag{4.2}$$

where

$$F(r) = \frac{3r^6 + (l^2 + 5n_2^2 + 10n_1^2)r^4 + 3(n_2^2l^2 + 10n_1^2n_2^2 + n_1^2l^2 + 5n_1^4)r^2}{3(r^2 + n_1^2)(r^2 + n_2^2)l^2}$$
$$+ \frac{6ml^2r - 3n_1^2n_2^2(l^2 + 5n_1^2)}{3(r^2 + n_1^2)(r^2 + n_2^2)l^2} \qquad (4.3)$$

Here the above metric is a solution of vacuum Einstein field equations with cosmological constant $(\lambda = -\frac{10}{l^2})$ if and only if $(n_1^2 - n_2^2)\lambda = 0$. Consequently we see that differing values for n_1 and n_2 are possible only if the cosmological constant vanishes. For $n_1 = n_2 = n$ the above solution reduces to the six-dimensional solution found in [18, 19]. In what follows we shall look at the case of two different nut charges, that is we set the cosmological constant to zero.

Let us consider the Euclidean section, obtained by the following analytic continuations $t \rightarrow i\chi$ and $n_j \rightarrow in_j$ where $j = 1, 2$:

$$ds^2 = F_E(r)(d\chi - 2n_1\cos\theta_1 d\varphi_1 - 2n_2\cos\theta_2 d\varphi_2)^2 + F_E^{-1}(r)dr^2$$
$$+ (r^2 - n_1^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2) \qquad (4.4)$$

where

$$F_E(r) = \frac{r^4 - 3(n_1^2 + n_2^2)r^2 + 6mr - 3n_1^2n_2^2}{3(r^2 - n_1^2)(r^2 - n_2^2)} \qquad (4.5)$$

This metric is a solution of the vacuum Einstein field equations without cosmological constant, for any values of the parameters n_1 and n_2. In order to analyze the possible singularities we have to consider two cases, depending on the values of the nut charges n_1 and n_2. Let us assume first that $n_1 = n_2 = n$. Then the Taub-nut solutions correspond to the fixed-point sets of $\frac{\partial}{\partial\chi}$ at $r = n$. The mass parameter $m = \frac{4n^3}{3}$, while the periodicity of the coordinate χ is $12\pi n$. For the bolt solution we impose the periodicity of the coordinate χ to be $\frac{12\pi n}{k}$. The fixed-point set is four-dimensional and located at $r = r_p = \frac{3n}{k} > n$, with the mass parameter given by $m = m_p = \frac{n^3(k^4 + 18k^2 - 27)}{6k^3}$. Hence we must consider only the values $k = 1, 2$, which in turn avoids the curvature singularity at $r = n$.

If the two nut charges n_1 and n_2 differ, we set $n_1 > n_2$ without loss of generality. In this case in the Euclidean section the radius r cannot be smaller than n_1 or the signature of the spacetime will change. The Taub-nut solution in this case corresponds to a two-dimensional fixed-point set located at $r = n_1$. There is still a curvature singularity located at $r = n_1$, removed by setting the periodicity of the coordinate χ to be $8\pi n_1$, while the value of the mass parameter must be $m = m_p = \frac{n_1^3 + 3n_1n_2^2}{3}$.

The bolt solution corresponds to a four-dimensional fixed-point set located at $r = r_b = \frac{2n_1}{k}$, for which the periodicity of the coordinate χ is given by $\frac{8\pi n_1}{k}$ and the value of the mass parameter is $m = m_p = \frac{n_1(12n_2^2 - 4n_1^2)}{12}$. In order to avoid the curvature singularity at $r = n_1$ we must choose $k = 1$ so that $r > r_b = 2n_1$.

In the following we consider the Taub-nut solution for which $n_1 > n_2$. The periodicity of the coordinate χ is taken to be $8\pi n_1$ while the value of the mass parameter is fixed to be $m = m_p = \frac{n_1^3 + 3n_1 n_2^2}{3}$. For these values the six-dimensional metric is nonsingular at $r = n_1$.

Employing the usual Kaluza-Klein procedure we obtain a six-dimensional magnetic monopole: we add a flat time direction to obtain a seven-dimensional solution of the vacuum Einstein field equations and after that perform a Kaluza-Klein compactification along the coordinate χ using the metric ansatz:

$$ds_7^2 = e^{\frac{\phi}{\sqrt{10}}} ds_6^2 + e^{-\frac{4\phi}{\sqrt{10}}} (d\chi - A_{(1)})^2$$

It is easy to check that we obtain the following six-dimensional fields

$$ds_6^2 = -F_E^{\frac{1}{4}} dt^2 + F_E^{-\frac{3}{4}} dr^2 + F_E^{\frac{1}{4}} \left[(r^2 - n_1^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2) \right]$$
$$A_{(1)} = -2n_1 \cos\theta_1 d\varphi_1 - 2n_2 \cos\theta_2 d\varphi_2, \qquad e^{-\frac{\phi}{\sqrt{10}}} = F_E^{\frac{1}{4}} \tag{4.6}$$

where we now restrict $r \geq n_1 > n_2$ and

$$F_E(r) = \frac{r^3 + n_1 r^2 - (2n_1^2 + 3n_2^2)r + 3n_1 n_2^2}{3(r + n_1)(r^2 - n_2^2)}$$

One can check that the above six-dimensional monopole solution has a curvature singularity located at $r = n_1$. It is interesting to note that the asymptotic structure of this solution, after rescaling the coordinates $t \to 3^{1/4}T$ and $r \to 3^{-3/8}R$ is given by

$$ds_{asymp}^2 = -dT^2 + dR^2 + \frac{1}{3}R^2(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + \frac{1}{3}R^2(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2)$$

The area of each 2-sphere is not $4\pi R^2$ but instead $\frac{4\pi R^2}{3}$: each has a deficit solid angle of $\frac{8\pi}{3}$ steradians. Furthermore, the above asymptotic form of the metric is Ricci flat, and can be obtained from our solution by setting $n = 0$. Therefore we conclude that the background for our monopole is a Ricci flat Bohm metric constructed as a cone over $S^2 \times S^2$ [20, 21].

The corresponding six-dimensional Lagrangian obtained after Kaluza-Klein reduction is given by

$$\mathcal{L}_6 = eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\frac{5}{\sqrt{10}}\phi}F_{(2)}^2$$

where $F_{(2)} = dA_{(1)} = 2n_1\Omega_1 + 2n_2\Omega_2$ and we have denoted by Ω_i the volume form $\sin\theta_i d\theta_i \wedge d\varphi_i$ of the sphere M_i, $i = 1, 2$.

In the following we shall perform a Kaluza-Klein reduction on M_2. The general sphere reduction formulae have been presented in [22]. The metric ansatz that we have to use in the dimensional reduction from six to four dimensions is given by:

$$ds_6^2 = e^{\frac{\varphi}{\sqrt{2}}} ds_4^2 + e^{-\frac{\varphi}{\sqrt{2}}}(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2)$$

The dimensionally-reduced Lagrangian will take now the form

$$\mathcal{L}_4 = eR - \frac{1}{2}e(\partial\varphi)^2 - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\frac{\varphi}{\sqrt{2}}-\frac{5}{\sqrt{10}}\phi}F^2 + ee^{\sqrt{2}\varphi}R_2 - e2n_2^2 e^{\frac{3}{\sqrt{2}}\varphi - \frac{5}{\sqrt{10}}\phi}$$

where $F = dA = d(-2n_1\cos\theta_1 d\phi_1)$ and $R_2 = 4$ is the Ricci scalar of the sphere M_2. The full solution in four dimensions will be given by:

$$ds_4^2 = -F_E^{\frac{1}{2}}(r^2 - n_2^2)dt^2 + F_E^{-\frac{1}{2}}(r^2 - n_2^2)dr^2 + F_E^{\frac{1}{2}}(r^2 - n_1^2)(r^2 - n_2^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2)$$
$$F = dA = 2n_1\sin\theta_1 d\theta_1 \wedge d\varphi_1, \qquad e^{-\frac{\phi}{\sqrt{10}}} = F_E^{\frac{1}{4}}, \qquad e^{-\frac{\varphi}{\sqrt{2}}} = F_E^{\frac{1}{4}}(r^2 - n_2^2) \qquad (4.7)$$

The asymptotic form of the above four-dimensional monopole metric is given by:

$$ds_{asymp}^2 \sim -Rdt^2 + dR^2 + \frac{4R^2}{3}(d\theta_1^2 + \sin^2\theta d\varphi_1^2) \qquad (4.8)$$

(after defining $r^2 = \frac{2R}{3^{\frac{1}{4}}}$ and rescaling the time coordinate t) and we can see that the spacetime is not asymptotically flat. Moreover the metric has infinite redshift at the origin[3], which is also the location of a curvature singularity. It takes an infinite time for a photon to reach infinity and, indeed, the (r,t)-sector is asymptotically flat; however, while the metric it is singularity-free at infinity the scalar field φ diverges there. It is interesting to note that the asymptotic form has a surfeit of solid angle, as the area of a sphere of radius R is not $4\pi R^2$ but $\frac{16\pi r^2}{3}$. Asymptotically conical metrics are reminiscent of global monopoles [16]. The magnetic charge is computed to be $2n_1$.

The second case to discuss in six dimensions is a generalization of the five-dimensional solution presented in the previous section. The metric ansatz is as follows:

$$ds^2 = -F(r)(dt - 2n\cos\theta_1 d\phi_1)^2 + F^{-1}(r)dr^2 + (r^2 + n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha r^2(d\theta_2^2 + d\phi_2^2)$$
$$F(r) = \frac{-3r^5 + (l^2 - 10n^2)r^3 + 3n^2(l^2 - 5n^2)r + 6ml^2}{3rl^2(r^2 + n^2)} \qquad (4.9)$$

and the vacuum Einstein field equations with cosmological constant are satisfied if and only if $\alpha(2 - \lambda n^2) = 0$. Since α cannot be zero we must restrict the values of n and $\lambda = \frac{10}{l^2}$ such

---

[3]A similar BPS monopole solution with an infinite redshift at the origin but with a deficit of solid angle has been obtained in [23].

that $\lambda n^2 = 2$, forcing a positive cosmological constant. The Euclidean section is obtained by taking the analytic continuation $t \to i\chi$ and $n \to in$ and $l \to il$ with $l = \sqrt{5}n$. Notice that in this case the Taub-nut solution corresponds to a two-dimensional fixed-point set of the vector field $\frac{\partial}{\partial \chi}$ located at $r = n$. The periodicity of the χ coordinate is in this case equal to $8\pi n$ and the value of the mass parameter is fixed to $m_b = \frac{n^3}{15}$. For this value of the mass parameter the solution is regular at the nut location. The bolt spacetime has a four-dimensional fixed-point set of $\frac{\partial}{\partial \chi}$ located at $r_b = \frac{kn}{2}$ and the value of the mass parameter is $m_b = \frac{k^3 n^3 (20 - 3k^2)}{960}$ where the periodicity of the coordinate χ is $\frac{8\pi n}{k}$, where k is an integer. To ensure that $r_b > n$ we have to take $k > 3$; in this way the curvature singularity at $r = n$ is avoided as well. We next perform another analytic continuation of one of the coordinates on T^2 (say $\theta_2 \to it$) and then two Kaluza-Klein reductions along the χ and ϕ_2 directions down to four dimensions. We obtain the final solution:

$$ds^2 = -r^3 F_E^{\frac{1}{2}} dt^2 + F_E^{-\frac{1}{2}}(r) r dr^2 + F_E^{\frac{1}{2}} r \left(r^2 - \frac{l^2}{5} \right) d\Omega^2$$

$$\mathcal{A} = -2n \cos\theta_1 d\phi_1, \qquad e^{\frac{-3\varphi_1}{\sqrt{6}}} = r^2, \qquad e^{\frac{-\varphi_2}{\sqrt{3}}} = r^{\frac{1}{3}} F_E^{-\frac{1}{2}} \qquad (4.10)$$

where

$$F_E(r) = \frac{15r^5 - 5l^2 r^3 + 30ml^2}{3l^2 r (5r^2 - l^2)}$$

which is a solution of the equations of motion derived from the following Lagrangean:

$$\mathcal{L}_4 = eR - \frac{1}{2} e(\partial \varphi_1)^2 - \frac{1}{2} e(\partial \varphi_2)^2 - \frac{1}{4} e e^{-\sqrt{3}\varphi_2} \mathcal{F}^2 + 2 e e^{\frac{\varphi_1}{\sqrt{6}} + \frac{\varphi_2}{\sqrt{3}}} \lambda$$

with $\lambda = -\frac{10}{l^2}$ and $\mathcal{F} = d\mathcal{A}$. The asymptotic form of the metric is:

$$ds_4^2 = -R^2 dt^2 + \frac{\sqrt{l}}{4R} dR^2 + R^2 d\Omega^2$$

after rescaling $R^2 = \frac{r^4}{l}$. It is clearly not asymptotically flat, it has infinite redshift at origin, which is also the location of a curvature singularity and as expected the dilaton fields diverge at infinity. However, notice that while φ_2 diverges at the root of F_E, φ_1 is finite there. The magnetic charge is found to be $2n$.

4.1.1 Hopf reductions in six dimensions

It is well-known that odd-dimensional spheres S^{2n+1} may be regarded as circle bundles over CP^n and one can use the so-called Hopf duality (a T-duality along the $U(1)$-fibre) to generate new solutions [24, 25, 26] by untwisting S^{2n+1} to $CP^n \times S^1$. The six-dimensional

case is particularly interesting for us since it has been shown in [25] that it is possible to make consistent truncations of the maximal Type II supergravity theories to a bosonic sector which exhibits an $O(2,2)$ global symmetry with the T-duality transformation taking a simple form. The theories at hand are the toroidal reductions of Type IIA, respectively Type IIB ten-dimensional supergravities while the reduction ansatz for the fields is that the six-dimensional fields that are retained are precisely the ten-dimensional ones, with the spacetime indices restricted to run over the six-dimensional range only. The two truncated theories in $D = 6$ are then related by a T-duality transformation upon reduction to $D = 5$. The explicit mappings of the fields have been given in [25] and we follow their notational conventions. For convenience we also provide the derivation of the T-duality rules in Appendix B.

Let us start with the solution given in (4.3) in which we set $\lambda = 0$. We shall perform first the analytic continuations [27] $t \to iz$, $n_1 \to in_1$ and subsequently $\varphi_2 \to it$:

$$ds^2 = \tilde{F}(r)(dz - 2n_1 \cos\theta_1 d\varphi_1 - 2n_2 \cos\theta_2 dt)^2 + \tilde{F}^{-1}(r)dr^2$$
$$+(r^2 - n_1^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + (r^2 + n_2^2)(d\theta_2^2 - \sin^2\theta_2 dt^2) \qquad (4.11)$$

where
$$\tilde{F}(r) = \frac{r^4 - 3(n_1^2 - n_2^2)r^2 + 6mr + 3n_1^2 n_2^2}{3(r^2 - n_1^2)(r^2 + n_2^2)}$$

Considering the above metric as a solution of the pure gravity sector of the truncated Type IIA theory we can now perform a Hopf-duality along the spacelike z-direction to obtain a solution of six-dimensional Type IIB theory:

$$ds_{6B} = \tilde{F}(r)^{-\frac{1}{2}}dz^2 + \tilde{F}(r)^{-\frac{1}{2}}dr^2 + \tilde{F}(r)^{\frac{1}{2}}(r^2 - n_1^2)d\Omega_1^2 + \tilde{F}(r)^{\frac{1}{2}}(r^2 + n_2^2)(d\theta_2^2 - \sin^2\theta_2 dt^2)$$
$$e^{2\phi_1} = e^{2\phi_2} = \tilde{F}(r), \qquad A_{(2)}^{NS} = -2n_1 \cos\theta_1 d\varphi_2 \wedge dz + 2n_2 \cos\theta_2 dt \wedge dz \qquad (4.12)$$

We can also make the analytic continuations $t \to i\varphi_2$ and $n_1 \to in_1$ to obtain the solution:

$$ds_{6B} = -F(r)^{-\frac{1}{2}}dt^2 + F(r)^{-\frac{1}{2}}dr^2 + F(r)^{\frac{1}{2}}(r^2 + n_1^2)d\Omega_1^2 + F(r)^{\frac{1}{2}}(r^2 + n_2^2)d\Omega_2^2$$
$$e^{2\phi_1} = e^{2\phi_2} = F(r), \qquad A_{(2)}^{NS} = 2n_1 \cos\theta_1 d\varphi_2 \wedge dt - 2n_2 \cos\theta_2 d\varphi_2 \wedge dt \qquad (4.13)$$

Were we to consider (4.11) as a solution of the pure gravity sector of Type IIB theory, then after performing the spacelike Hopf dualisation we would obtain as an intermediate step

$$ds_{6A} = \tilde{F}(r)^{-\frac{1}{2}}dz^2 + \tilde{F}(r)^{-\frac{1}{2}}dr^2 + \bar{F}(r)^{\frac{1}{2}}(r^2 - n_1^2)d\Omega_1^2 + \bar{F}(r)^{\frac{1}{2}}(r^2 + n_2^2)(d\theta_2^2 - \sin^2\theta_2 dt^2)$$
$$e^{2\phi_1} = e^{2\phi_2} = \bar{F}(r), \qquad A_{(2)} = -2n_1 \cos\theta_1 d\varphi_2 \wedge dz + 2n_2 \cos\theta_2 dt \wedge dz \qquad (4.14)$$

15

Performing the analytic continuations we recover (4.13) except that we have now to replace $A_{(2)}^{NS}$ with $A_{(2)}$.

It is interesting to note that we can perform directly a timelike Hopf reduction in six-dimensions[4]. In this case if we start with a solution of Type IIA theory by performing the timelike T-duality we obtain a solution of the appropriate truncation of Type IIB* theory [28]. If we start instead with a solution of Type IIB theory and perform a timelike Hopf duality we end up with a solution of an appropriate truncation of Type IIA* theory. The details of these reductions are gathered in the Appendix B.

As an example we shall perform a Hopf duality starting from Type IIA theory. Consider (4.3) as a solution of the pure gravity sector of the truncated six-dimensional Type IIA theory. Then the final solution of Type IIB* will be given by:

$$ds_{6B^*} = -F(r)^{-\frac{1}{2}}dt^2 + F(r)^{-\frac{1}{2}}dr^2 + F(r)^{\frac{1}{2}}(r^2 + n_1^2)d\Omega_1^2 + F(r)^{\frac{1}{2}}(r^2 + n_2^2)d\Omega_2^2$$

$$e^{2\phi_1} = e^{2\phi_2} = F(r), \qquad A_{(2)}^{NS} = 2n_1 \cos\theta_1 d\varphi_1 \wedge dt - 2n_2 \cos\theta_2 d\varphi_2 \wedge dt \qquad (4.15)$$

where now

$$F(r) = \frac{r^4 - 3(n_1^2 + n_2^2)r^2 + 6mr - 3n_1^2 n_2^2}{3(r^2 + n_1^2)(r^2 + n_2^2)}$$

If we start with (4.3) as a solution of Type IIB then peforming a timelike Hopf dualisation we obtain a similar solution of Type IIA* for which:

$$A_{(2)} = 2n_1 \cos\theta_1 d\varphi_1 \wedge dt - 2n_2 \cos\theta_2 d\varphi_2 \wedge dt \qquad (4.16)$$

As we can see, some of the solutions obtained for Type IIA (respectively IIB) and Type IIA* (respectively IIB*) are identical after we perform appropriate analytic continuations[5]. This is to be expected once we notice that they are solutions of the $NSNS$-sector only, which is the same for both theories (their actions would differ only by the sign of the kinetic terms of the RR-fields).

As an application of these solutions let us set for convenience $n_2 = 0$ in (4.13) and perform a sphere reduction using the ansatz (4.1) down to a four-dimensional solution:

$$ds_{4B} = -r^2 dt^2 + r^2 dr^2 + F(r)r^2(r^2 + n_1^2)d\Omega_1^2$$

$$e^{-\sqrt{2}\phi} = r^4 F(r), \qquad e^{2\phi_1} = e^{2\phi_2} = F(r), \qquad A_{(2)}^{NS} = 2n_1 \cos\theta_1 d\varphi_2 \wedge dt \qquad (4.17)$$

which is a solution of the equations of motion derived from the following Lagrangian:

$$\mathcal{L}_{4B} = eR - \frac{1}{2}e(\partial\varphi)^2 - \frac{1}{2}e(\partial\phi_1)^2 - \frac{1}{2}e(\partial\phi_2)^2 - \frac{1}{4}ee^{-\frac{\varphi}{\sqrt{2}}-\varphi_1-\varphi_2}(F_{(3)}^{NS})^2 + ee^{\sqrt{2}\varphi}R_2$$

___

[4]In which case we do not need to perform any analytical continuations.

[5]Which keep the metric and the fields real.

The asymptotic form of the metric (4.17) is (after defining $R = \frac{r^2}{2}$ and rescaling t)

$$ds_{4B}^2 \sim -Rdt^2 + dR^2 + \frac{4}{3}R^2 d\Omega_1^2$$

Amusingly, the asymptotic form of the metric is the same with (4.8). The magnetic charge is found to be $2n$ and notice that there is an excess of solid angle as the area of the asymptotic sphere is $\frac{16\pi R^2}{3}$ instead of the expected $4\pi R^2$.

4.2 Monopoles in $D \geq 7$ dimensions

Similarly, we can construct Kaluza-Klein monopoles in seven and higher dimensions. For example, in seven dimensions the base space is five-dimensional and can be factorized in the form $B = M \times Y$, where M is an even dimensional space endowed with an Einstein-Kähler metric and Y is a Riemannian Einstein space.

Let us consider the case in which $M = S^2$ while Y can be a sphere S^3, a torus T^3 or a hyperboloid H^3. The solution is [5]

$$ds^2 = -F(r)(dt^2 + 2n\cos\theta d\varphi)^2 + F^{-1}(r)dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + \beta r^2 g_Y$$

where g_Y is the metric on the unit-sphere S^3, torus T^3 or hyperboloid H^3:

$$F(r) = \frac{4r^6 + (l^2 + 12n^2)r^4 + 2n^2(l^2 + 6n^2)r^2 + 4ml^2 + n^4(l^2 + 6n^2)}{4l^2 r^2 (r^2 + n^2)} \tag{4.18}$$

The cosmological constant is $\lambda = -\frac{15}{l^2}$ and the parameters β, n and λ are constrained via the relation $\beta(5 - 2\lambda n^2) = 10k$, where $k = 1, 0, -1$ for S^3, T^3 and H^3 respectively. We must have $\beta > 0$, which in turn imposes a joint constraint on λn^2 that can be satisfied in various ways depending on the value of k. Since we are interested in a Ricci flat solution we shall consider $\lambda = 0$ and also $k = 1$, in which case $Y = S^3$ and $\beta = 2$. The Euclidean section of this solution, which is obtained by analytic continuation of the coordinate $t \to i\chi$ and of the parameter $n \to in$ is given by:

$$ds^2 = F_E(r)(d\chi^2 + 2n\cos\theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + 2r^2 g_Y$$

where

$$F(r) = \frac{r^4 - 2n^2 r^2 + n^4 + 4m}{4r^2(r^2 - n^2)} \tag{4.19}$$

The metric has a scalar curvature singularity located at $r = n$ as can be checked by computing for instance the Kretschman scalar. However, if we take $m = 0$ then the metric is well-behaved at $r = n$. Furthermore, if $r < n$ the signature of the metric is unphysical; therefore

17

we can restrict ourselves to the interval $r \geq n$ for which the solution is non-singular. In order to obtain the magnetic brane in seven-dimensions we employ the usual procedure: add a flat timelike direction and compactify the Ricci-flat eight-dimensional metric using the ansatz

$$ds_8^2 = e^{\frac{\phi}{\sqrt{15}}} ds_7^2 + e^{-\frac{5\phi}{\sqrt{15}}} (d\chi - A_{(1)})^2$$

We obtain the following seven-dimensional fields:

$$ds_7^2 = -F_E^{\frac{1}{5}} dt^2 + F_E^{-\frac{4}{5}} dr^2 + F_E^{\frac{1}{5}} \left((r^2 - n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + 2r^2 d\Omega_3^2 \right)$$
$$A_{(1)} = -2n\cos\theta d\varphi, \qquad e^{-\frac{\phi}{\sqrt{15}}} = F_E^{\frac{1}{5}} \tag{4.20}$$

where now $r \geq n$ and

$$F_E(r) = \frac{r^2 - n^2}{4r^2}$$

which are a solution of the equations of motion derived from the following Lagrangian:

$$\mathcal{L}_7 = eR - \frac{1}{2} e(\partial\phi)^2 - \frac{1}{4} e e^{-\frac{6}{\sqrt{15}}\phi} F_{(2)}^2$$

The above seven-dimensional metric has a curvature singularity at $r = n$. Its asymptotic structure, after we rescale the coordinates $t \rightarrow 4^{1/10} T$ and $r \rightarrow 4^{-2/5} R$, is given by

$$ds_{asymp}^2 = -dT^2 + dR^2 + \frac{R^2}{4}(d\theta^2 + \sin^2\theta d\varphi^2) + 2^{3/5} R^2 d\Omega_3^2$$

This space has a deficit of solid angle corresponding to the sphere S^2 while the factor S^3 has a surfeit of solid angle.

Let us perform now a further dimensional reduction of the above seven-dimensional solution on the three-sphere S^3. The metric ansatz that we can use in the dimensional reduction from 7 to 4 dimensions is given by:

$$ds_7^2 = e^{\frac{3\varphi}{2\sqrt{15}}} ds_4^2 + e^{-\frac{\varphi}{\sqrt{15}}} d\Omega_3^2$$

The four-dimensional fields will be given by

$$ds_4^2 = 2^{\frac{3}{2}} r^3 \left(-F_E^{\frac{1}{2}} dt^2 + F_E^{\frac{1}{2}} dr^2 + F_E^{\frac{1}{2}} (r^2 - n^2)(d\theta^2 + \sin^2\theta d\varphi^2) \right)$$
$$A_{(1)} = -2n\cos\theta d\varphi, \qquad e^{-\frac{\phi}{\sqrt{15}}} = F_E^{\frac{1}{5}}, \qquad e^{-\frac{\varphi}{\sqrt{15}}} = 2r^2 F_E^{\frac{1}{5}} \tag{4.21}$$

and they are a solution of the equations of motion derived from the following dimensionally reduced Lagrangian:

$$\mathcal{L}_4 = eR - \frac{1}{2} e(\partial\varphi)^2 - \frac{1}{2} e(\partial\phi)^2 - \frac{1}{4} e e^{-\frac{3\varphi}{2\sqrt{15}} - \frac{6}{\sqrt{15}}\phi} F_{(2)}^2 + e e^{\frac{5}{\sqrt{15}}\varphi} R_3$$

18

where $R_3 = 6$ is the curvature scalar of the unit sphere S^3.

One can check that the above four-dimensional solution has a scalar curvature singularity at $r = n$. Its asymptotics are given by:

$$ds^2 = R^3 \left[-dT^2 + dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) \right]$$

after we make the rescaling $R = \sqrt{2}r$ and $T = t/2$. Consequently the spacetime that we obtain is conformally flat and singularity free at infinity. The magnetic charge is found to be $2n$.

Generalization to more than seven dimensions is straightforward. For instance, in eight dimensions we can consider a circle fibration over the product $S^2 \times S^2 \times S^2$ and associate for each sphere factor a nut charge n_i, $i = 1..3$. We obtain the following metric

$$ds^2 = -F(r)(dt + \mathcal{A})^2 + \frac{dr^2}{F(r)} + \sum_{i=1,3}(r^2 + n_i^2)d\Omega_i^2$$

$$F(r) = \frac{3r^6 + 5(n_1^2 + n_2^2 + n_3^2)r^4 + 15(n_1^2 n_2^2 + n_2^2 n_3^2 + n_1^2 n_3^2)r^2 - 15mr - 15n_1^2 n_2^2 n_3^2}{15(r^2 + n_1^2)(r^2 + n_2^2)(r^2 + n_3^2)}$$

$$\mathcal{A} = 2n_1\cos\theta_1 d\phi_1 + 2n_2\cos\theta_2 d\phi_2 + 2n_3\cos\theta_3 d\phi_3 \tag{4.22}$$

that is a natural generalization of the eight-dimensional solution presented in [19]. Our metric contain three different nut parameters. Taking its Euclidean section (obtained by analytically continuing $t \to i\chi$, $n_j \to in_j$ for $j = 1..3$) and adding a trivial time direction we can then perform Kaluza-Klein reductions along the χ direction and also along two of the spheres to obtain a final four-dimensional monopole solution.

We can also consider different factorizations of the base space in terms of products of Einstein-Kähler manifolds with various lower dimensional Einstein spaces (as in [5]).

Another interesting solution can be found in eleven dimensions by using the ansatz:

$$ds^2 = -F(r)(dt + 2n\cos\theta d\phi)^2 + \frac{dr^2}{F(r)} + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2) + 6r^2 d\Omega_7^2$$

and by solving the vacuum Einstein field equations we find:

$$F(r) = \frac{3r^8 + 4n^2 r^6 + 24m}{24r^6(r^2 + n^2)}$$

Here $d\Omega_7^2$ is the metric on the 7-sphere, normalized such that its Ricci tensor is $R_{ij} = 6g_{ij}$. The Euclidean section is obtained by analytic continuations $t \to i\chi$ and $n \to in$. We can also

replace the sphere element by any other Einstein space of positive curvature. For example, if we embed the seven dimensional de Sitter solution we obtain the metric:

$$ds^2 = \tilde{F}(r)(d\chi + 2n\cos\theta d\phi)^2 + \frac{dr^2}{\tilde{F}(r)} + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$+r^2\left[-\left(1 - \frac{R^2}{6}\right)dt^2 + \frac{dR^2}{1 - \frac{R^2}{6}} + R^2 d\Omega_5^2\right]$$

$$\tilde{F}(r) = \frac{3r^8 - 4n^2 r^6 + 24m}{24r^6(r^2 - n^2)} \tag{4.23}$$

It can be easily checked there is a curvature singularity located at $r = 0$. Misner string singularities can be removed by requiring the χ coordinate to have period $8\pi n$ and $m = \frac{n^8}{24}$. The values of the coordinate r are then restricted to $r > n$ avoiding the curvature singularity at $r = 0$. This solution corresponds to a seven dimensional fixed-point set of the isometry ∂_χ; since this is not the maximal possible co-dimension, it is a Taub-nut solution.

The other possibility is that of a nine-dimensional fixed-point set of the isometry ∂_χ, located at $r_b = 2n$. The periodicity of the χ coordinate is still $8\pi n$ but now the values of the r coordinate are such that $r \geq r_b = 2n$. This in turn avoids the curvature singularities located at $r = 0$ and $r = n$, provided the value of the mass parameter is $m = -\frac{64n^8}{3}$.

Let us perform now a Kaluza-Klein compactification along the coordinate χ. The reduction ansatz is:

$$ds_{11}^2 = e^{\frac{\varphi}{6}}ds_{10}^2 + e^{-\frac{4\varphi}{3}}(d\chi + \mathcal{A})^2$$

and we obtain the following ten-dimensional fields:

$$ds_{10}^2 = \tilde{F}^{\frac{1}{8}}(r)r^2\left[-\left(1 - \frac{R^2}{6}\right)dt^2 + \frac{dR^2}{1 - \frac{R^2}{6}} + R^2 d\Omega_5^2\right]$$

$$+\tilde{F}^{-\frac{7}{8}}dr^2 + \tilde{F}^{\frac{1}{8}}(r)(r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\mathcal{A} = 2n\cos\theta d\phi, \qquad e^{-\frac{4\varphi}{3}} = \tilde{F}(r) \tag{4.24}$$

Now let us perform a sphere reduction of this solution down to five-dimensions using the metric ansatz:

$$ds_{10}^2 = e^{\sqrt{\frac{5}{12}}\phi}ds_5^2 + e^{-\sqrt{\frac{3}{20}}\phi}d\Omega_5^2 \tag{4.25}$$

We obtain the following fields:

$$ds_{5A} = -\tilde{F}^{\frac{1}{3}}r^{\frac{16}{3}}R^{\frac{10}{3}}\left(1 - \frac{R^2}{6}\right)dt^2 + \tilde{F}^{\frac{1}{3}}r^{\frac{16}{3}}R^{\frac{10}{3}}\frac{dR^2}{1 - \frac{R^2}{6}} + \tilde{F}^{-\frac{2}{3}}r^{\frac{10}{3}}R^{\frac{10}{3}}dr^2 + \tilde{F}^{\frac{1}{3}}R^{\frac{10}{3}}r^{\frac{10}{3}}(r^2 - n^2)d\Omega_2^2$$

$$\mathcal{A} = 2n\cos\theta d\phi, \qquad e^{-\frac{4\varphi}{3}} = \tilde{F}(r), \quad e^{-\sqrt{\frac{3}{20}}\phi} = \tilde{F}^{\frac{5}{24}}r^{\frac{10}{3}}R^{\frac{10}{3}} \tag{4.26}$$

20

which give a solution of the equations of motion derived from the Lagrangian:

$$\mathcal{L}_5 = eR - \frac{1}{2}e(\partial\varphi)^2 - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\frac{3\varphi}{2}-\sqrt{\frac{5}{12}}\phi}(\mathcal{F}_{(2)})^2 + ee^{\frac{4}{\sqrt{15}}\phi}R_5$$

where R_5 is the curvature scalar of the unit 5-sphere. We can dualize $\mathcal{F}_{(2)}$ to a 3-form field strength and we find that the above solution would describe a non-uniform electric string in five dimensions as our solution depends explicitly on the fifth dimension R. This solution is very likely to be unstable as in eleven dimensions the 'de Sitter horizon' is delocalised along the noncompact direction r.

5   Conclusions

In this paper we have constructed higher dimensional Kaluza-Klein brane solutions with and without cosmological constant, generalizing the original KK-monopole solution [1, 2]. The simplest generalization in four dimensions uses the Taub-bolt solution as a seed and the physical interpretation of the final KK-bolt monopole solution was recently clarified in the literature [12, 13]: it corresponds to a pair of coincident extremal dilatonic black holes with opposite unequal magnetic charges.

In five dimensions we considered cosmological Taub-NUT-like metrics [5, 6] and performed similar KK reductions down to four dimensions. The new feature of these solutions is that the four-dimensional dilaton acquires a potential term in the Lagrangian as an effect of the cosmological constant. However their asymptotics are not very appealing physically since they are not asymptotically flat or asymptotically $(A)dS$. Their metric description simplifies when considered in the string frame. For our explicit examples the four-dimensional metric in the string frame is very similar to the AdS form in the (r, t) sector, except for a deficit of solid angle in the angular sector, which is characteristic for global monopoles. Another feature of the above constructions is that in five dimensions and in some of the higher dimensional examples the nut charge and the cosmological constant are intimately related by a constraint equation imposed by the equations of motion. This constraint makes it impossible to consider situations in which either the cosmological constant or the nut charge go to zero. From this perspective the above monopole solutions are qualitatively distinct from their predecessors.

On the other hand, in six and seven dimensions we have considered non-singular Ricci-flat solutions for which one can use the KK trick to obtain similar KK magnetic brane solutions for which the background spaces are Ricci flat Bohm spaces of the form $S^p \times S^q$

and generically have conical singularities. We considered their further reduction down to four dimensions on Riemannian spaces of constant curvature and specifically considered such reductions on spheres. In higher dimensions, we showed that we can also consider monopole solutions in cosmological backgrounds. All the above solutions from section 4 have a corresponding extension for which the cosmological constant is non-zero and are similarly expected to produce magnetic KK brane solutions upon dimensional reduction. In contrast with the KK procedure to untwist the $U(1)$-fibration, we have considered in six dimensions another method, which is known to untwist the circle fibration, namely Hopf duality in string theory. We extended the Hopf duality rules to the case of a timelike Hopf-duality of the truncated six-dimensional Type II theories and applied them to generate charged string solutions in six-dimensions. By performing sphere reductions we obtained the corresponding four-dimensional solutions with magnetic charges.

As avenues for further research it would be interesting to use the recently discovered nut-charged rotating solutions in higher dimensions [31]. It is known that starting from the four dimensional NUT-charged Kerr solution, the KK procedure leads to a solution describing a brane/anti-brane pair carrying opposite unbalanced magnetic charges [12]. The case of higher dimensional Kerr solution has been explored in [32] where it was found that for instance in six dimensions one obtains a string loop instead of a pair of monopoles and anti-monopoles. It would be interesting to see the effect of the nut charge in these cases. Work on this is in progress and will be reported elsewhere.

Acknowledgements

A: Kaluza-Klein Reduction

The usual metric ansatz for the dimensional reduction from $(D+1)$ dimensions to D dimensions in Kaluza-Klein theory is given by:

$$d\hat{s}^2 = e^{2\alpha\varphi}ds^2 + e^{-2(D-2)\alpha\varphi}(dz + \mathcal{A})^2 \tag{A-1}$$

with $\alpha = \sqrt{\frac{1}{2(D-1)(D-2)}}$. Here the D-dimensional metric ds^2 corresponds to the so-called Einstein frame, while its conformal rescaling $e^{2\alpha\varphi}ds^2$ is the metric in the string frame.

Then a general bosonic Lagrangian in $(D+1)$ dimensions of the form:

$$\mathcal{L} = \hat{e}\hat{R} + 2\hat{e}\lambda - \frac{1}{2}\hat{e}(\partial\hat{\phi})^2 - \frac{1}{2n!}\hat{e}e^{\hat{a}\hat{\phi}}\hat{F}_n^2$$

will reduce in D dimensions to the Lagrangian given by the formula [29]:

$$\mathcal{L} = eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{2}e(\partial\varphi)^2 + 2e\lambda e^{2\alpha\varphi} - \frac{1}{4}e \cdot e^{-2(D-1)\alpha\varphi}\mathcal{F}^2$$
$$-\frac{1}{2n!}e \cdot e^{-2(n-1)\alpha\varphi + \hat{a}\phi}F'^2_n - \frac{1}{2(n-1)!}e \cdot e^{2(D-n)\alpha\varphi + \hat{a}\phi}F^2_{n-1} \qquad \text{(A-2)}$$

Here $e = \sqrt{-g}$, $\mathcal{F} = d\mathcal{A}$ and $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ is the one-form potential that appears in the Kaluza-Klein form of the metric, while the ansatz used for the KK-reduction of the matter fields from $(D+1)$-dimensions is $\hat{\phi} = \phi$ for the scalar field, and $\hat{A}_{n-1} = A_{n-1} + A_{n-2} \wedge dz$ for the anti-symmetric potential $\hat{A}_{n-1}$. The field strengths of the potentials A_{n-1} and A_n are $F_n = dA_{n-1}$, respectively $F_{n-1} = dA_{n-2}$ and we have defined $F'_n = F_n - F_{n-1} \wedge \mathcal{A}$.

Note that the presence of the cosmological constant in the higher dimensional theory induces a scalar potential for the Kaluza-Klein scalar field φ. Also, if the isometry generated by the Killing vector $\frac{\partial}{\partial z}$ has fixed points, then the dilaton φ will diverge and the D-dimensional metric will be singular at those points.

B: T-duality in six dimensions

The Lagrangian in $D = 6$ obtained by dimensional reduction of Type IIB on a torus and after performing a consistent truncation is given by [25]:

$$\mathcal{L}_{6B} = eR - \frac{1}{2}e(\partial\phi_1)^2 - \frac{1}{2}e(\partial\phi_2)^2 - \frac{1}{2}ee^{2\phi_1}(\partial\chi_1)^2 - \frac{1}{2}ee^{2\phi_2}(\partial\chi_2)^2$$
$$-\frac{1}{12}ee^{-\phi_1-\phi_2}(F_{(3)}^{NS})^2 - \frac{1}{12}ee^{\phi_1-\phi_2}(F_{(3)}^{RR})^2 + \chi_2 dA_{(2)}^{NS} \wedge dA_{(2)}^{RR} \qquad \text{(B-1)}$$

where $F_{(3)}^{NS} = dA_{(2)}^{NS}$ and $F_{(3)}^{RR} = dA_{(2)}^{RR} + \chi_1 dA_{(2)}^{NS}$. This Lagrangian is related by T-duality in $D = 5$ to a different six-dimensional theory obtained by making a consistent truncation of Type IIA compactified on a four-dimensional torus. The corresponding Lagrangian is given by:

$$\mathcal{L}_{6A} = eR - \frac{1}{2}e(\partial\phi_1)^2 - \frac{1}{2}e(\partial\phi_2)^2 - \frac{1}{48}ee^{\frac{\phi_1}{2}-\frac{3\phi_2}{2}}(F_{(4)})^2 - \frac{1}{12}ee^{-\phi_1-\phi_2}(F_{(3)})^2$$
$$-\frac{1}{4}ee^{\frac{3\phi_1}{2}-\frac{\phi_2}{2}}(F_{(2)})^2 \qquad \text{(B-2)}$$

where $F_{(4)} = dA_{(3)} - dA_{(2)} \wedge A_{(1)}$, $F_{(3)} = dA_{(2)}$ corresponds to the NS-NS 3-form $F_{(3)1}$ and $F_{(2)} = dA_{(1)}$ is the RR 2-form $\mathcal{F}_{(2)}^1$, with the index '1' denoting here the first reduction step from $D = 11$ to $D = 10$.

Let us focus on Type IIA theory first. Under a dimensional reduction using the formulae from the previous appendix we have:

$$ds_6^2 = e^{\frac{\varphi}{\sqrt{6}}} ds_5^2 + e^{\frac{-3\varphi}{\sqrt{6}}} (dz + \mathcal{A}_{(1)})^2 \tag{B-3}$$

and we obtain the following 5-dimensional Lagrangian:

$$\mathcal{L}_{5A} = eR - \frac{1}{2} e(\partial\phi_1)^2 - \frac{1}{2} e(\partial\phi_2)^2 - \frac{1}{2} e(\partial\varphi)^2 - \frac{1}{48} e e^{-\frac{3\varphi}{\sqrt{6}} + \frac{\phi_1}{2} - \frac{3\phi_2}{2}} (F'_{(4)})^2$$

$$-\frac{1}{12} e e^{\frac{\varphi}{\sqrt{6}} + \frac{\phi_1}{2} - \frac{3\phi_2}{2}} (F_{(3)1})^2 - \frac{1}{12} e e^{-\frac{2\varphi}{\sqrt{6}} - \phi_1 - \phi_2} (F'_{(3)})^2 - \frac{1}{2} e e^{-\frac{4\varphi}{\sqrt{6}}} \mathcal{F}_{(2)}^2$$

$$-\frac{1}{4} e e^{-\frac{\varphi}{\sqrt{6}} + \frac{3\phi_1}{2} - \frac{\phi_2}{2}} (F'_{(2)})^2 - \frac{1}{4} e e^{\frac{2\varphi}{\sqrt{6}} - \phi_1 - \phi_2} (F_{(2)1})^2 - \frac{1}{2} e e^{\frac{3\varphi}{\sqrt{6}} + \frac{3\phi_1}{2} - \frac{\phi_2}{2}} (dA_{(0)1})^2 \tag{B-4}$$

where the field strengths are defined as follows (see the formulae from appendix A):

$$F'_{(2)} = dA_{(1)} - dA_{(0)1} \wedge \mathcal{A}_{(0)}, \qquad F'_{(3)} = dA_{(2)} - dA_{(1)} \wedge \mathcal{A}_{(1)}$$

$$F_{(3)1} = dA_{(2)1} + dA_{(1)} \wedge A_{(1)} - dA_{(2)} \wedge A_{(0)1}, \qquad F'_{(4)} = dA_{(3)} - dA_{(2)} \wedge A_{(1)} - F_{(3)1} \wedge \mathcal{A}_{(1)}$$

while $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ and $F_{(2)1} = dA_{(1)1}$. Upon dualising $F_{(4)}$ to a 1-form field strength $d\chi'$ its kinetic term in the above Lagrangian will be replaced by:

$$-\frac{1}{2} e e^{\frac{3\varphi}{\sqrt{6}} - \frac{\phi_1}{2} + \frac{3\phi_2}{2}} (d\chi')^2 + \chi' F'_{(3)} \wedge F'_{(2)} + \chi' F_{(3)1} \wedge \mathcal{F}_{(2)} \tag{B-5}$$

If we perform the field redefinitions field redefinitions:

$$A'_{(1)} = A_{(1)} - A_{(0)1} \wedge \mathcal{A}_{(1)}, \quad A'_{(2)} = A_{(2)} - A_{(1)1} \wedge \mathcal{A}_{(1)}, \quad A'_{(2)1} = A_{(2)1} + A_{(1)1} \wedge A'_{(1)} \tag{B-6}$$

we find:

$$F'_{(2)} = dA'_{(1)} + A_{(0)1} \wedge \mathcal{F}_{(2)}, \qquad F'_{(3)} = dA'_{(2)} - A_{(1)1} \wedge \mathcal{F}_{(2)}$$

$$F_{(3)1} = dA'_{(2)1} + dA'_{(1)} \wedge A_{(1)1} - A_{(0)1}(dA'_{(2)} - A_{(1)1} \wedge \mathcal{F}_{(2)})$$

$$\chi' F'_{(3)} \wedge F'_{(2)} + \chi' F_{(3)1} \wedge \mathcal{F}_{(2)} = \chi'(dA'_{(2)} \wedge dA'_{(1)} + dA'_{(2)1} \wedge \mathcal{F}_{(2)}) \tag{B-7}$$

Similarly, for the dimensional reduction of Type IIB Lagrangian we obtain:

$$\mathcal{L}_{5B} = eR - \frac{1}{2} e(\partial\phi_1)^2 - \frac{1}{2} e(\partial\phi_2)^2 - \frac{1}{2} e(\partial\varphi)^2 - \frac{1}{2} e e^{2\phi_1} (\partial\chi_1)^2 - \frac{1}{2} e e^{2\phi_2} (\partial\chi_2)^2$$

$$-\frac{1}{12} e e^{-\frac{2\varphi}{\sqrt{6}} + \phi_1 - \phi_2} (F'^{RR}_{(3)})^2 - \frac{1}{12} e e^{-\frac{2\varphi}{\sqrt{6}} - \phi_1 - \phi_2} (F'^{NS}_{(3)})^2 - \frac{1}{2} e e^{-\frac{4\varphi}{\sqrt{6}}} \mathcal{F}_{(2)}^2 - \frac{1}{4} e e^{\frac{2\varphi}{\sqrt{6}} + \phi_1 - \phi_2} (F^{RR}_{(2)1})^2$$

$$-\frac{1}{4} e e^{\frac{2\varphi}{\sqrt{6}} - \phi_1 - \phi_2} (F^{NS}_{(2)1})^2 - \chi_2 dA^{RR}_{(2)} \wedge dA^{NS}_{(1)1} + \chi_2 dA^{NS}_{(2)} \wedge dA^{RR}_{(1)1} \tag{B-8}$$

where $F^{NS}_{(2)1} = dA^{NS}_{(1)1}$, $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ and:

$$F'^{NS}_{(3)} = dA^{NS}_{(2)} - dA^{NS}_{(1)1} \wedge \mathcal{A}_{(1)}, \qquad F^{RR}_{(2)1} = dA^{RR}_{(1)1} + \chi_1 dA^{NS}_{(1)1}$$

$$F'^{RR}_{(3)} = dA^{RR}_{(2)} - dA^{RR}_{(1)1} \wedge \mathcal{A}_{(1)} + \chi_1 dA^{NS}_{(2)} \wedge \mathcal{A}_{(1)} \tag{B-9}$$

As shown first in [25], the T-duality rules relating the two truncated theories (B-4) and (B-8) are:

$$A_{(0)1} \to \chi_1, \quad A_{(1)1} \to \mathcal{A}_{(1)}, \quad \mathcal{A}_{(1)} \to A^{NS}_{(1)1}, \quad \chi' \to \chi_2,$$

$$A'_{(1)} \to A^{RR}_{(1)1}, \quad A'_{(2)} \to A^{NS}_{(2)}, \quad A'_{(2)1} \to -A^{RR}_{(2)} \tag{B-10}$$

together with a rotation of the scalars:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \varphi \end{pmatrix}_{IIA,B} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \varphi \end{pmatrix}_{IIB,A} \tag{B-11}$$

which takes care of the dilaton couplings of the field strengths.

We are also interested in performing a timelike T-duality. As it is known this duality will relate Type IIA (respectively IIB) to Type IIB* (respectively IIA*). We wish to see if at the level of our truncated theories the timelike T-duality rules are still valid.

Consider first the Type IIA theory. Upon a timelike dimensional reduction the Lagrangian of the reduced theory will have a form similar with (B-4); however the kinetic terms for $F_{(3)1}$, $F_{(2)1}$, $\mathcal{F}_{(2)}$ and $dA_{(0)1}$ will have the reversed sign [30]. The five-dimensional metric is now of Euclidean signature and when we dualize the 4-form $F'_{(4)}$ to a scalar field strength χ' we obtain a positive kinetic term for this scalar. We expect to be able to relate this theory to a timelike reduction of a truncated six-dimensional Type IIB* theory by applying the T-duality rules given above. Now, it is known that the action of Type IIB* in ten dimensions is obtained from the usual Type IIB action after we reverse the signs of the RR kinetic terms. As the sign of such kinetic terms was irrelevant when discussing the truncation to six dimensions we see that a consistent truncation of Type IIB* in six dimensions will be given by the Lagrangian (B-1) in which we must reverse the sign on the kinetic terms for the RR fields, i.e. we must reverse the sign of the kinetic terms for $F^{RR}_{(3)}$ and also for χ_2 (which appears from the dualisation of the RR field $B_{(4)}$). When performing a timelike dimensional reduction the final Type IIB* Lagrangian will be similar with (B-8) with reverted signs for the kinetic terms of χ_2, $F^{RR}_{(3)}$, $F^{NS}_{(2)1}$ respectively $\mathcal{F}_{(2)}$. It is then straightforward to see that the T-duality will relate our truncated Type IIA theory with the truncated Type IIB*. It is

25

easy to extend the above considerations to show that a timelike T-duality will relate Type IIB with Type IIA* at the level of our truncated theories. Also Type IIA* and Type IIB* are related by a usual T-duality along a spacelike direction.

References

[1] Rafael D. Sorkin "Kaluza-Klein Monopole" Phys. Rev. Lett. B **51** (1983) 87.

[2] D. J. Gross and M. J. Perry "Magnetic Monopoles in Kaluza-Klein theory" Nucl. Phys. **B226** (1983) 29.

[3] V. K. Onemli and B. Tekin, "Kaluza-Klein monopole in AdS spacetime," [arXiv:hep-th/0301027].

[4] D. Astefanesei, R. B. Mann and E. Radu, "Nut charged space-times and closed timelike curves on the boundary," [arXiv:hep-th/0407110]; D. Astefanesei and E. Radu "Quantum effects in a rotating spacetime", Int. J. Mod. Phys. D **11** 715 (2002) [arXiv:gr-qc/0112029].

[5] R. Mann and C. Stelea,"Nuttier (A)dS black holes in higher dimensions," Class. Quant. Grav. **21**, 2937 (2004), [arXiv:hep-th/0312285].

[6] H. Lu, D. N. Page and C. N. Pope, "New inhomogeneous Einstein metrics on sphere bundles over Einstein-Kaehler manifolds," [arXiv:hep-th/0403079].

[7] R. Clarkson, R. B. Mann, "Eguchi-Hanson Solitons", [arXiv:hep-th/00508109]; R. Clarkson, R. B. Mann, "Eguchi-Hanson Solitons in Odd Dimensions" , [arXiv:hep-th/0508200].

[8] C. W. Misner "The Flatter Regions of Newman, Unti and Tamburino's Generalized Schwarzschid Space" J. Math. Phys. **4** (1963) 924.

[9] D. N. Page "Taub-NUT instanton with a Horizon" Phys. Lett. B **78** (1978) 249-251.

[10] R.B. Mann, "Misner String Entropy", Phys. Rev. D **60**, 104047 (1999) [arXiv:hep-th/9903229].

[11] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, "Large N phases, gravitational instantons and the NUTs and bolts of AdS holography," Phys. Rev. D **59**, 064010 (1999) [arXiv:hep-th/9808177].

[12] Y. C. Liang and E. Teo, "Black diholes with unbalanced magnetic charges," Phys. Rev. D **64**, 024019 (2001) [arXiv:hep-th/0101221].

[13] A. Chamblin and R. Emparan, "Bubbles in Kaluza-Klein theories with space- or time-like internal dimensions," Phys. Rev. D **55**, 754 (1997) [arXiv:hep-th/9607236].

[14] M. Demiansky and E. T. Newman, "A combined Kerr-NUT solution of the Einstein field equations" Bull. Acad. Polon. Sci. Ser. Math. Astron. Phys. **14** 653(1966).

[15] G. W. Gibbons and M. J. Perry, "New Gravitational Instantons And Their Interactions," Phys. Rev. D **22**, 313 (1980).

[16] M. Barriola and A. Vilenkin, "Gravitational Field Of A Global Monopole," Phys. Rev. Lett. **63**, 341 (1989).

[17] J. F. Sparks,"Kaluza-Klein branes," [arXiv:hep-th/0105209].

[18] A. Awad and A. Chamblin, "A bestiary of higher dimensional Taub-NUT-AdS space-times," Class. Quant. Grav. **19**, 2051 (2002) [arXiv:hep-th/0012240].

[19] F. A. Bais and P. Batenberg, "A New Class of Higher-Dimensional Kaluza-Klein Monopole and Instanton Solutions", Nucl. Phys. B **253**, 162 (1985).

[20] C. Bohm,"Inhomogenous Einstein metrics on low-dimensional spheres and other low-dimensional spaces," Invent. Math. **134**, 145 (1998).

[21] G. W. Gibbons, S. A. Hartnoll and C. N. Pope, "Bohm and Einstein-Sasaki metrics, black holes and cosmological event horizons," Phys. Rev. D **67**, 084024 (2003) [arXiv:hep-th/0208031].

[22] M. S. Bremer, M. J. Duff, H. Lu, C. N. Pope and K. S. Stelle, "Instanton cosmology and domain walls from M-theory and string theory," Nucl. Phys. B **543**, 321 (1999) [arXiv:hep-th/9807051].

[23] S. A. Hartnoll,"Axisymmetric non-abelian BPS monopoles from G(2) metrics," Nucl. Phys. B **631**, 325 (2002) [arXiv:hep-th/0112235].

[24] M. J. Duff, H. Lu and C. N. Pope, "AdS(5) x S(5) untwisted," Nucl. Phys. B **532**, 181 (1998) [arXiv:hep-th/9803061].

[25] M. J. Duff, H. Lu and C. N. Pope, "AdS(3) x S**3 (un)twisted and squashed, and an O(2,2,Z) multiplet of dyonic strings," Nucl. Phys. B **544**, 145 (1999) [arXiv:hep-th/9807173].

[26] M. Cvetic, H. Lu and C. N. Pope, "Consistent warped-space Kaluza-Klein reductions, half-maximal gauged supergravities and CP(n) constructions," Nucl. Phys. B **597**, 172 (2001) [arXiv:hep-th/0007109].

[27] D. Astefanesei and G. C. Jones, "S-branes and (anti-)bubbles in (A)dS space", [arXiv:hep-th/0502162]; D. Astefanesei, R. B. Mann and C. Stelea,"Nuttier bubbles," [arXiv:hep-th/0508162].

[28] C. M. Hull, "Timelike T-duality, de Sitter space, large N gauge theories and topological field theory," JHEP **9807**, 021 (1998) [arXiv:hep-th/9806146].

[29] H. Lu, C. N. Pope, E. Sezgin and K. S. Stelle, "Stainless super p-branes," Nucl. Phys. B **456**, 669 (1995) [arXiv:hep-th/9508042].

[30] E. Cremmer, I. V. Lavrinenko, H. Lu, C. N. Pope, K. S. Stelle and T. A. Tran, "Euclidean-signature supergravities, dualities and instantons," Nucl. Phys. B **534**, 40 (1998) [arXiv:hep-th/9803259].

[31] D. Klemm, "Rotating black branes wrapped on Einstein spaces," JHEP **9811**, 019 (1998) [arXiv:hep-th/9811126].

[32] B. Janssen and S. Mukherji, "Kaluza-Klein dipoles, brane/anti-brane pairs and instabilities," [arXiv:hep-th/9905153].